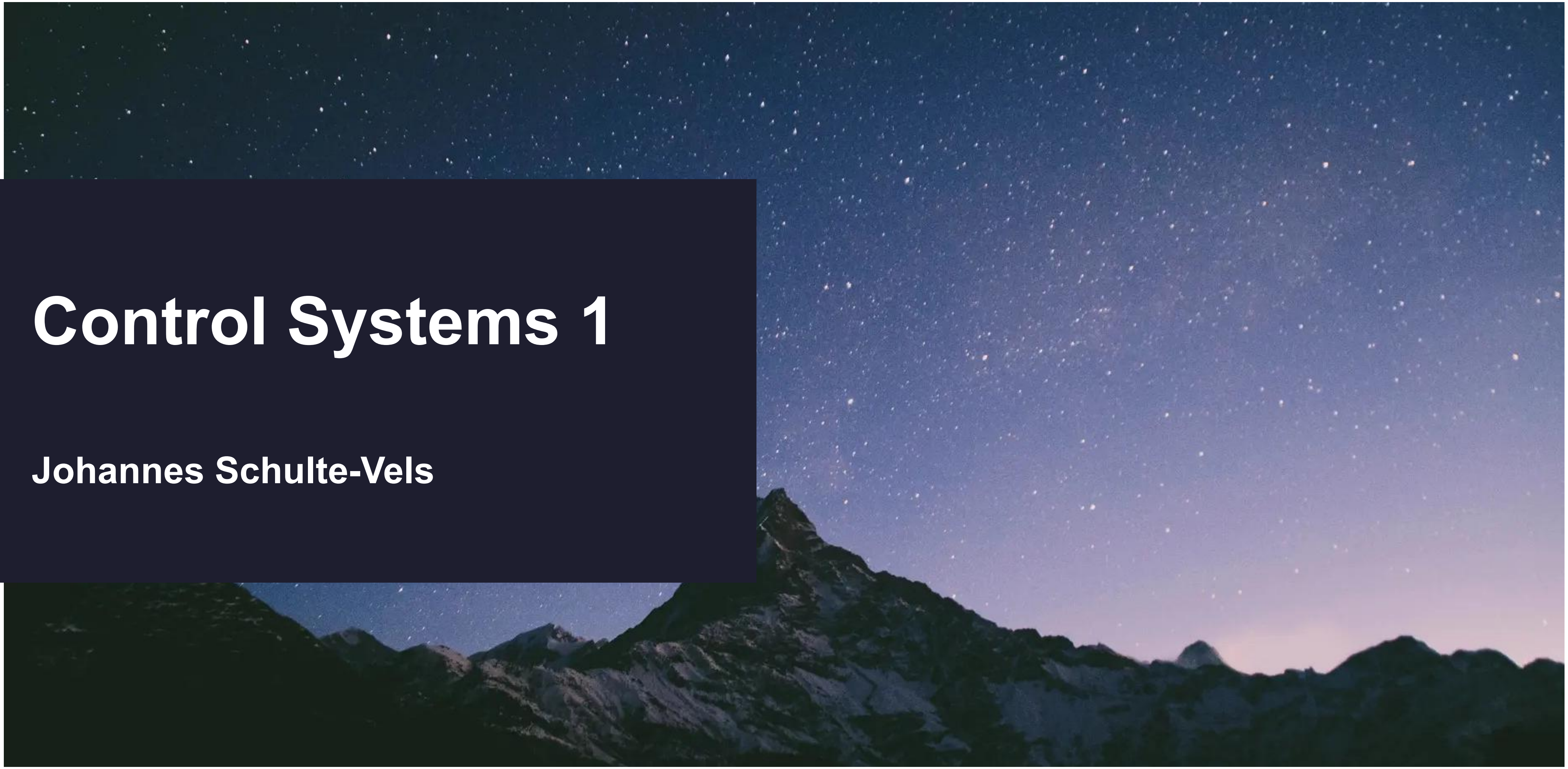
ETH zürich

Institute for Dynamic Systems and Control



Control Systems 1

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Welcome!

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Today

- Repetition Session 2
- Theory Recap
  - System Classification
  - LTI State-space Model
  - Linearization
- Important Note!
- Q&A Session / Done

Repetition Session 2

Modeling

We want to mathematically represent our dynamic system. All those system can be represented using ODE’s. By convention, we express those in the state space form.

$$
\begin{aligned}
\dot{x}(t) &= f(x(t), u(t)) \\
y(t) &= h(x(t), u(t))
\end{aligned}
$$

x(t) is a vector of all the state variables, describing how the system changes internally over time. So called memory.

HS 2018

$y = a + b + c$

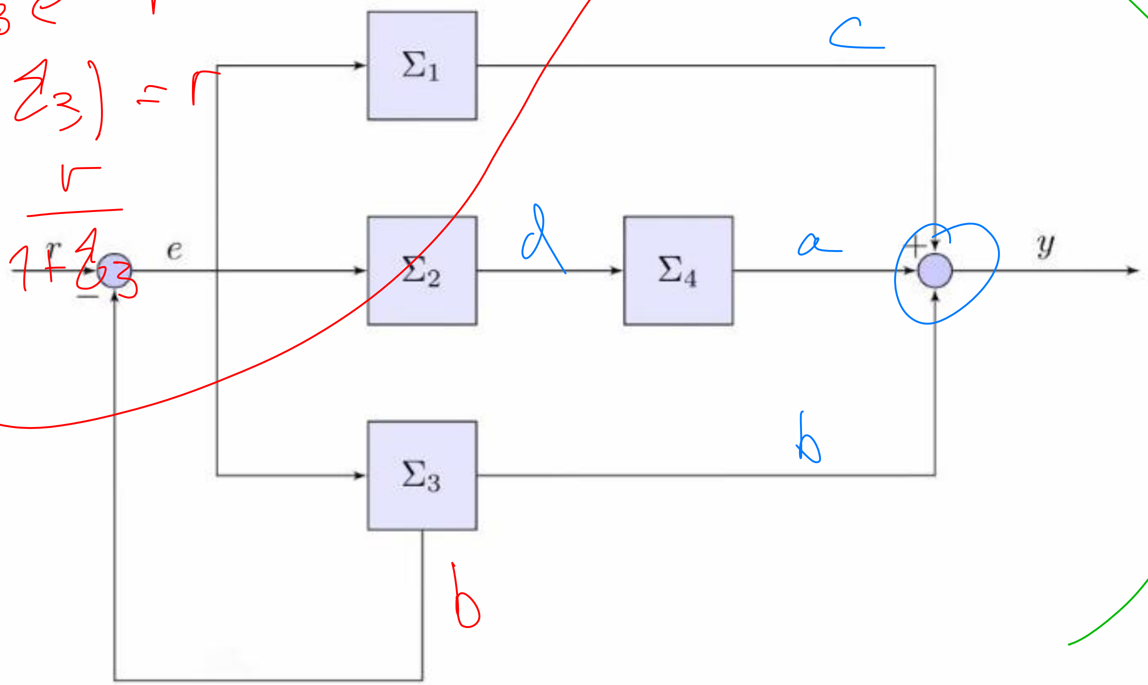
$= e \cdot (\Sigma_4 \Sigma_2 + \Sigma_3 + \Sigma_1) = \frac{1}{1+\Sigma_3} \cdot (\Sigma_1 + \Sigma_2 \Sigma_4 + \Sigma_3) \cdot r$

$e = r - b = r - \Sigma_3 \cdot e$

$e + \Sigma_3 e = r$

$e(1+\Sigma_3) = r$

$e = \frac{r}{1+\Sigma_3}$



A) $\Sigma_1 + \Sigma_2 \Sigma_4 + \Sigma_3$

B) $\frac{\Sigma_1 + \Sigma_2 \Sigma_4 + \Sigma_3}{1 + \Sigma_2 \Sigma_4}$

C) $\frac{\Sigma_1 + \Sigma_2 \Sigma_4 + \Sigma_3}{1 + \Sigma_3}$

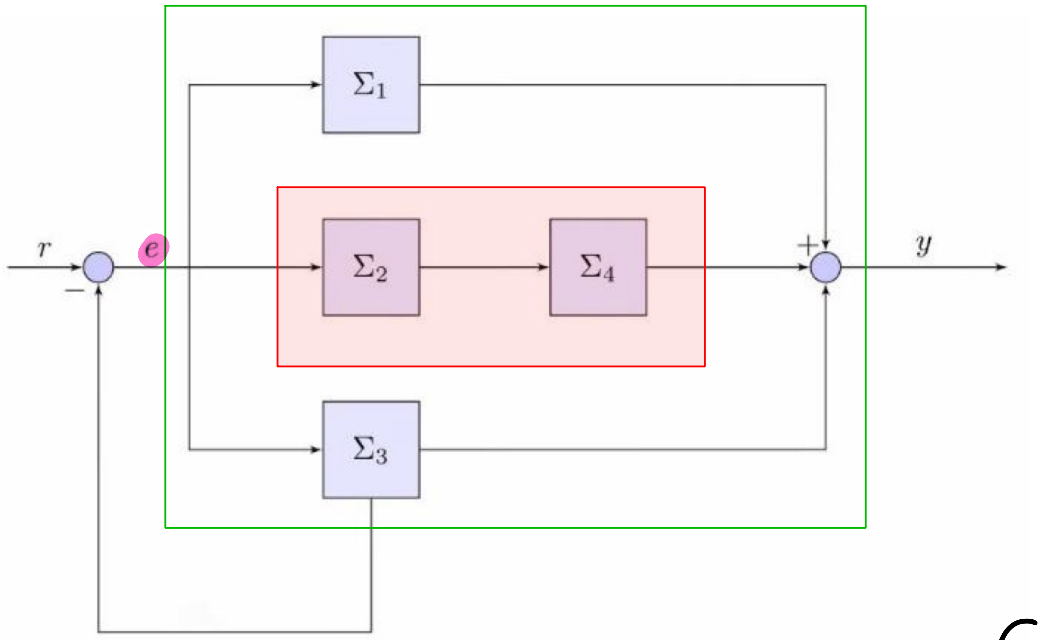
D) $\frac{\Sigma_1 + \Sigma_2 \Sigma_4 + \Sigma_3}{1 + \Sigma_1 + \Sigma_2 \Sigma_4 + \Sigma_3}$

$b = \Sigma_3 e$

$c = \Sigma_1 e$

$\left.\begin{aligned} a &= \Sigma_4 d \\ d &= \Sigma_2 e \end{aligned}\right\} \quad a = \Sigma_4 \Sigma_2 e$

Annotations



$$\text{(red box)} = \Sigma_2 \Sigma_4$$

$$\text{(green box)} = \Sigma_1 + \text{(red box)} + \Sigma_3$$

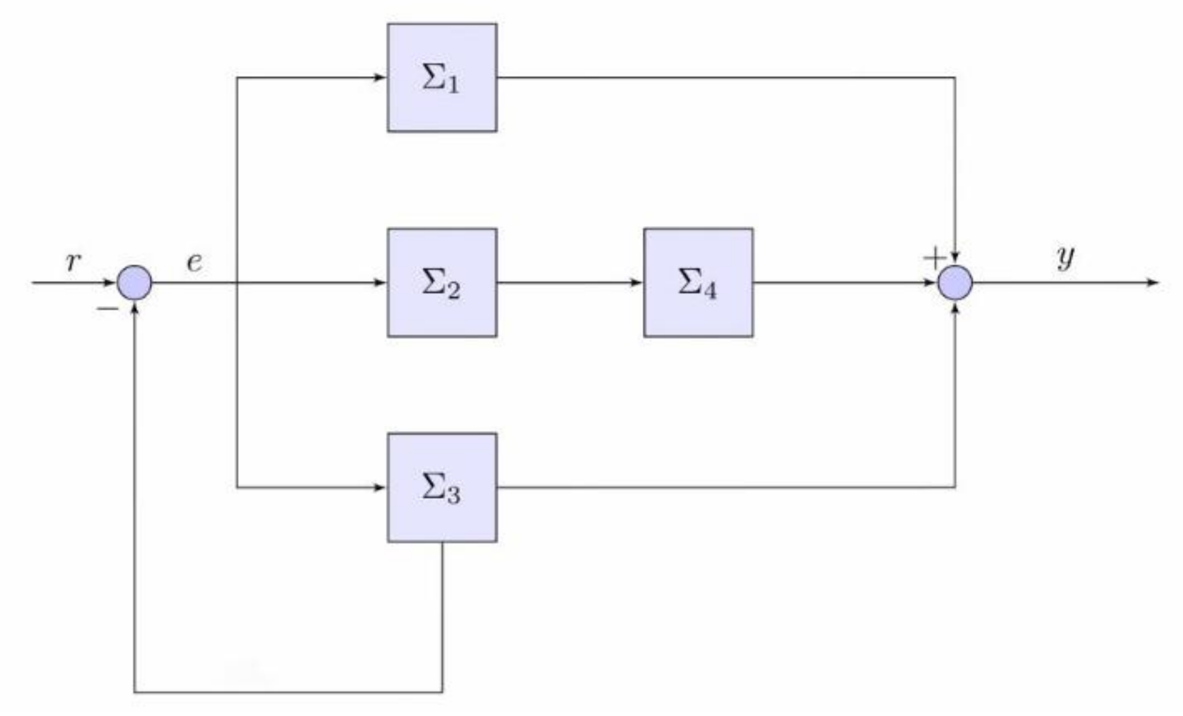
$$e = r - \Sigma_3 \cdot e \Rightarrow (1 + \Sigma_3) e = r$$

$$\Rightarrow e = \frac{1}{1+\Sigma_3} \cdot r$$

Combine:

$$y = \boxed{\frac{\Sigma_1 + \Sigma_2\Sigma_4 + \Sigma_3}{1 + \Sigma_3}} \cdot r$$

HS 2018



A) $\Sigma_1 + \Sigma_2\Sigma_4 + \Sigma_3$

B) $\frac{\Sigma_1+\Sigma_2\Sigma_4+\Sigma_3}{1+\Sigma_2\Sigma_4}$

C) $\frac{\Sigma_1+\Sigma_2\Sigma_4+\Sigma_3}{1+\Sigma_3}$

D) $\frac{\Sigma_1+\Sigma_2\Sigma_4+\Sigma_3}{1+\Sigma_1+\Sigma_2\Sigma_4+\Sigma_3}$

Theory Recap

System Classification

System Classification

With the modeling approach introduced we are able to model a large variety of systems.

Now we can also classify them in different ways. Because in this course we will only cover so called LTI (Linear, Time Invariant) Systems, this classification is important to us.

Generally, there are the following classifications:

- **Linear** or **Nonlinear**
- **Causal** or **Noncausal**
- **Static** (memoryless) or **Dynamic**
- **Time invariant** or **Time variant**

Linear or Nonlinear

For a System Σ to be linear, we need to have the following properties:

1. Additivity: $\Sigma(u_1 + u_2) = \Sigma u_1 + \Sigma u_2$
2. Homogeneity: $\Sigma(\alpha u) = \alpha \Sigma u \;,\; \alpha \in \mathbb{R}$

$$\frac{x(t+h) - x(t)}{h}$$

Summarized, this leads to the following:

$$\Sigma(\alpha u_1 + \beta u_2) = \alpha \Sigma u_1 + \beta \Sigma u_2, \quad \alpha, \beta \in \mathbb{R}.$$

Examples of ***linear*** systems:

- Integrator: $y(t) = \int_{-\infty}^{t} u(\tau) d\tau$
- Derivative: $y(t) = \dot{u}(t)$
- Time shifts: $y(t) = u(t - \tau)$
- Time scaling: $y(t) = u(t^2)$

Examples of ***non-linear*** systems:

- $y(t) = u(t)^2$
- $y(t) = u(t) + a$
- $y(t) = \sin(u(t))$

Causal or Noncausal

A system is called casual, **iff** (if and only if) the output depends only on past and current inputs, but never on future inputs (future does not change the present).

Only causal systems are physically realizable

$u \in (-\infty, t]$

Causal systems include:

- $y(t) = u(t)$
- $y(t) = u(t-\tau), \ \forall \tau > 0$
- $y(t) = \cos(3t+1)u(t-1)$
- $y(t) = \int_{-\infty}^{t} u(\tau)d\tau$

Non-Causal systems include:

- $y(t) = u(t-a), \ \forall a < 0$
- $y(t) = u(t+1) + u(t) + u(t-1)$
- $y(t) = u(bt), \ \forall b > 1$
- $y(t) = \int_{-\infty}^{t+1} u(\tau)d\tau$

Static (memoryless) or Dynamic

In static systems, the output only depends on the current input (no past, no future)

Memoryless (or static) systems include:

- $y(t) = 3u(t)$
- $y(t) = t^2 u(t)$
- $y(t) = 2^{-(t+1)} u(t)$
- $y(t) = \sqrt{\sin(u^2(t))}$

Dynamic systems include:

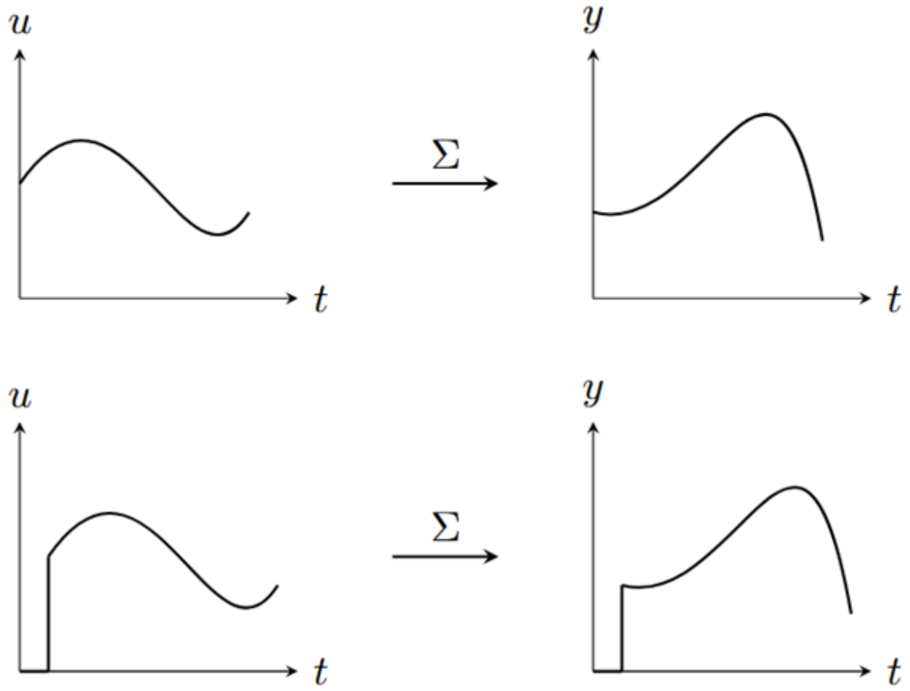
- $y(t) = \int_{-\infty}^{t} u(\tau) d\tau$
- $y(t) = \dot{u}(t)$
- $y(t) = u(t^2)$
- $y(t) = u(t-a), \ \forall a \neq 0$

Time invariant or Time variant

A time invariant system will always have the same output to a certain input, independent of when the input is applied. We can shift the input in time and the output will also be shifted by the same amount.

Example: An ideal lightbulb will always generate the same amount of light, no matter if at night or during the day.

Counterexample: A rocket losing mass (fuel) along it's way and being exposed to less gravitational force. The same thrust (input) leads to a different distance moved (output).



General rule of thumb:The system is time-invariant if the system equations do not contain time t as a summand, factor or exponent. Time t only appears in u(t).

Mathematically: $$y(t-\tau) = (\Sigma\tilde{u})(t), \text{ where } \tilde{u}(t) = u(t-\tau).$$

Time varying systems include:

- $y(t) = \cos(t)u(t)$
- $y(t) = u(3t+1)$
- $y(t) = \sqrt{t - \cos^2(t)}\; u(t^2)$
- $y(t) = \int_{-\infty}^{t-2} e^{-t} u(\tau + 2) d\tau$

Time invariant systems include:

- $y(t) = u(t-1)u(t+2)$
- $y(t) = 1 + u(t-1) + 2u(t)$
- $y(t) = 2\int_{-\infty}^{t} u^2(\tau) d\tau$
- $y(t) = \cos(u(t)) \int_{-\infty}^{t-2} u(\tau) d\tau$

Specific Example Time-invariancy

$$y(t) = \sqrt{t - \cos^2(t)} \cdot u(t^2)$$

Remember what we want to check: $y(t-\tau) = \Sigma(u(t-\tau))$

$$\textbf{LHS:}\ y(t-\tau) = \sqrt{(t-\tau) - \cos^2(t-\tau)} \cdot u\big((t-\tau)^2\big)$$

$$\begin{aligned}\textbf{RHS:}\ \Sigma u(t-\tau) = \Sigma\tilde{u}(t) &= \sqrt{t - \cos^2(t)} \cdot \tilde{u}\big(t^2\big) \\ &= \sqrt{t - \cos^2(t)} \cdot u\big((t-\tau)^2\big)\end{aligned}$$

$$\textbf{LHS} \neq \textbf{RHS} \rightarrow \textbf{Time-variant}$$

FS 2024

$$\Sigma_1 : \quad y(t) = \frac{1}{4}t + u(t-2).$$

A) Linear

C) Time-invariant

B) Causal

D) Static

FS 2024

$\alpha = 2$

$\Sigma(\alpha u) \stackrel{?}{=} \alpha \Sigma u$

$\frac{1}{4}t + 2u(t-2) \stackrel{?}{=} 2 \cdot \left(\frac{1}{4}t + u(t-2)\right)$

$$\Sigma_1 : \quad y(t) = \frac{1}{4}t + u(t-2).$$

extra term

extra t - term

A) Linear

C) Time-invariant

B) Causal

D) Static

no $u(t^*)$ for any $t^* > t$

$u(t-2)$

FS 2024

$$\Sigma_2: \quad y(t) = 3u(t) + \int_{-\infty}^{t} \left(1 + e^{-1}\right)^2 u(\tau)\mathrm{d}\tau.$$

A) Linear

C) Time-invariant

B) Causal

D) Static

Linear:

$\alpha \cdot \Sigma u \stackrel{?}{=} \Sigma(\alpha u)$, set $\alpha = 2$

$2 \cdot \left(\frac{1}{4}t + u(t-2)\right) \neq \frac{1}{4}t + 2 \cdot u(t-2)$

FS 2024

$u(\tau - \tau^*)\, d(\tau - \tau^*)$

$$\Sigma_2: \quad y(t) = 3u(t) + \int_{-\infty}^{t} \left(1 + e^{-1}\right)^2 u(\tau)\mathrm{d}\tau.$$

- no terms without u
- only linear u

tedious calc.

rule of thumb!

A) Linear

C) Time-invariant

B) Causal

D) Static

no $u(t^*)$ for any $t^* > t$

$\int_{-\infty}^{t} \dots d\tau$

LTI State-space Model

State-Space Model

As already mentioned, we will only look at LTI systems. More precisely, LTI SISO systems. (We omit the «C» for Causal because it sounds bad…?)

- Single Input, Single Output (SISO)
- Linear
- Time invariant
- Causal

Cool thing is, for those special systems we can rewrite our known state spacel model in a different form:

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = h(x(t), u(t)).$$

$$\Rightarrow$$

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t),$$

State-space Model

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t),$$

- A, B, C and D are **general matrices**.
- The order of the system is defined by the dimension of x(t)

For a LTI SISO system with $x \in \mathbb{R}^n$

$$A \in \mathbb{R}^{n \times n}, \; B \in \mathbb{R}^{n \times 1}, \; C \in \mathbb{R}^{1 \times n}, \; D \in \mathbb{R}$$

Example

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Given the following system, put it into LTI State-space form:

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -\frac{k}{m}x_1(t) + \frac{1}{m}u(t).$$
$$y(t) = x_2(t).$$

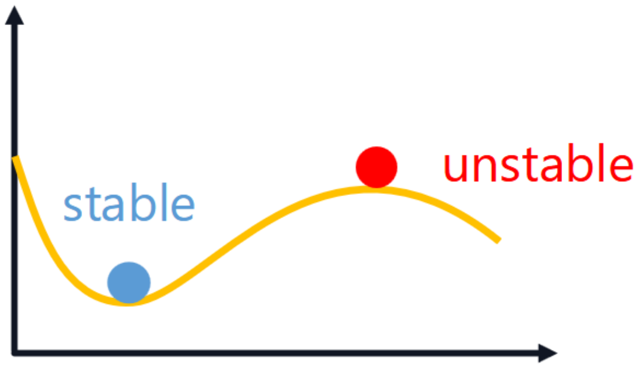
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) + 0 \quad u(t).$$

Linearization

Linearization

Unfortunately, most real life system are not linear and thus not representable in the LTI form. Luckily we can approximate using **linearization**.

To do so, we need to find an **equilibrium point**, around which we can linearize the system. An equilibrium point is some combination of state and input where the system is static → does not change.



Definition (Equilibrium point)

A system described by an ODE $\dot{x}(t) = f(x(t), u(t))$ has an equilibrium point (x_e, u_e) if

$$f(x_e, u_e) = 0.$$

Linearization

So the **general procedure** looks like this:

1. Find equilibrium point by solving $f(x_e, u_e) = 0$.
2. Linearize around the equilibrium point using Jacobian-Linearization-Procedure, which is basically Taylors-series. More precisely, our matrices will look like this:

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{(x_e,u_e)} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{(x_e,u_e)}$$

$$B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{(x_e,u_e)} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \vdots \\ \frac{\partial f_n}{\partial u} \end{bmatrix}\right|_{(x_e,u_e)}$$

$$C = \left.\frac{\partial h(x,u)}{\partial x}\right|_{(x_e,u_e)} = \left.\begin{bmatrix} \frac{\partial h}{\partial x_1} & \cdots & \frac{\partial h}{\partial x_n} \end{bmatrix}\right|_{(x_e,u_e)}$$

$$D = \left.\frac{\partial h(x,u)}{\partial u}\right|_{(x_e,u_e)} = \left.\begin{bmatrix} \frac{\partial h}{\partial u} \end{bmatrix}\right|_{(x_e,u_e)}$$

FS 2022

Aufgabe: Gegeben ist das folgende nichtlineare System mit Zustandsvektor $x(t)$ und Eingang $u(t)$.
Die Systemgleichungen lauten:

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 2 \cdot u(t) + 3 \cdot x_1(t) \\ 2 \cdot x_3(t) \\ -x_2(t) - 5 + \sqrt{u(t)} \end{bmatrix}.$$

Der Systemausgang ist gegeben durch

$$y(t) = x_3(t) + u(t).$$

F11 **(1 Punkt)** Berechnen Sie den reellen Gleichgewichtspunkt $x_{1,e}$, $x_{2,e}$, $x_{3,e}$ für den Ausgang $y_e = 1$.

$x_{1,e} =$

$x_{2,e} =$

$x_{3,e} =$

FS 2022

Aufgabe: Gegeben ist das folgende nichtlineare System mit Zustandsvektor $x(t)$ und Eingang $u(t)$.
Die Systemgleichungen lauten:

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 2 \cdot u(t) + 3 \cdot x_1(t) \\ 2 \cdot x_3(t) \\ -x_2(t) - 5 + \sqrt{u(t)} \end{bmatrix} .$$

3
1
4

Der Systemausgang ist gegeben durch

$$y(t) = x_3(t) + u(t) .$$

2

F11 **(1 Punkt)** Berechnen Sie den reellen Gleichgewichtspunkt $x_{1,e}$, $x_{2,e}$, $x_{3,e}$ für den Ausgang $y_e = 1$.

$$x_{1,e} = -\frac{2}{3}$$

$$x_{2,e} = -4$$

$$x_{3,e} = 0$$

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = f(\underline{x_e}, u_e)$$

1. $\Rightarrow$ $x_{3,e} = 0$
2. $\Rightarrow$ $u_e = 1$
3. $\Rightarrow$ $x_1 = -\frac{2}{3}$
4. $\Rightarrow$ $x_2 = -4$

FS 2022

Aufgabe: Gegeben ist das folgende nichtlineare System mit Zustandsvektor $x(t)$ und Eingang $u(t)$.

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \cdot x_2(t) + u^2(t) \\ -x_1(t) + 3 \end{bmatrix} .$$

Der Systemausgang ist gegeben durch

$$y(t) = x_1(t) - u(t) .$$

Linearisieren Sie das System um den Punkt

$$x_0 = \begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} ,$$

$$u_0 = 1 .$$

Annotations

$$A = \frac{\partial f(x,u)}{\partial x}\bigg|_{(x_e,u_e)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\Bigg|_{(x_e,u_e)}$$

$$B = \frac{\partial f(x,u)}{\partial u}\bigg|_{(x_e,u_e)} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \vdots \\ \frac{\partial f_n}{\partial u} \end{bmatrix}\Bigg|_{(x_e,u_e)}$$

$$C = \frac{\partial h(x,u)}{\partial x}\bigg|_{(x_e,u_e)} = \begin{bmatrix} \frac{\partial h}{\partial x_1} & \cdots & \frac{\partial h}{\partial x_n} \end{bmatrix}\bigg|_{(x_e,u_e)}$$

$$D = \frac{\partial h(x,u)}{\partial u}\bigg|_{(x_e,u_e)} = \begin{bmatrix} \frac{\partial h}{\partial u} \end{bmatrix}\bigg|_{(x_e,u_e)}$$

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \cdot x_2(t) + u^2(t) \\ -x_1(t) + 3 \end{bmatrix}.$$

Der Systemausgang ist gegeben durch

$$y(t) = x_1(t) - u(t).$$

Linearisieren Sie das System um den Punkt

$$x_0 = \begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$
$$u_0 = 1.$$

$$A = \begin{bmatrix} x_2 & x_1 \\ -1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 2u \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$D = -1$$

$$A = \begin{bmatrix} x_2 & x_1 \\ -1 & 0 \end{bmatrix}\Bigg|_{(x_E,u_E)}$$

$$\Rightarrow A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$$

FS 2022

F12 (1 Punkt)

$A =$

F13 (0.5 Punkte)

$b =$

F14 (0.5 Punkte)

$c =$	$d =$

FS 2022

F12 **(1 Punkt)**

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$$

F13 **(0.5 Punkte)**

$$b = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

F14 **(0.5 Punkte)**

$$c = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad d = \begin{bmatrix} -1 \end{bmatrix}$$

Important Note

Tips for the Exam?? (Personal Take)

- As already mentioned, this years exam will be different, because you will be given a summary sheet. Further, you will be allowed to bring your own 40 Page **handwritten** notes.

- Why do I tell you this:
  You are able to vote on Moodle what the summary sheet should contain. Use this!
  As a reference of which topics are important, I will upload 2 nice summaries (but they are ofc not handwritten).

- **Highly recomended**: Use the uploaded summary sheets (or any you find online) and solve exercises with them. You will see what information you actually need.
  Start now with your potential 40 page notes, as you will forget stuff as the semester advances. Put general solving strategies / examples and or weird twists on these pages and store them somewhere safe. You won't have the time or nerves to do this during the Lernphase!

Feedback

https://n.ethz.ch/~jschul/Feedback

Q&A Session / Done